ELSEVIER
Topology and its Applications 122 (2002) 625–636
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

Relative separation properties

Paul M. Gartside [a,*], Aneirin Glyn [b]

[a] *Department of Mathematics, University of Pittsburgh, Pittsburg, PA 15260, USA*
[b] *Department of Mathematics, St. Hugh's College, Oxford, UK OX2 6LE*

Received 30 November 2000; received in revised form 26 February 2001

Abstract

One positive answer and several counter-examples to problems posed in *Some Questions and References on Relative Topological Properties, Part 1* [Topology Atlas, June 15, 2000] by A.V. Arhangel'skii are presented. © 2002 Elsevier Science B.V. All rights reserved.

AMS classification: Primary 54D20, Secondary 54A35

Keywords: Relative separation property; Relative covering property

Introduction

In this note we present a variety of counter-examples regarding relative separation properties. The examples solve problems formulated by A.V. Arhangel'skii in *Some Questions and References on Relative Topological Properties, Part 1*, Topology Atlas website, June 15, 2000 [2]. (All numbered problems below are from Arhangelskii's Problem Collection, unless otherwise stated.) More particularly, Problems 1, 2, 3, 4, 5, 6 (consistently), 7, 9, 10, 17 and 19 are settled. The reader is referred to [1] for a survey of relative topological properties, and the motivation for Arhangel'skii's questions. Most of the relative properties defined in this paper were introduced by Arhangel'skii, and are contained in his survey. Problem 30 from [1] is also given a consistent answer.

Typical of the problems tackled here is the following (Problem 1):

Let Y be a subspace of a Hausdorff space X such that every closed subset of X contained in Y is compact (in short, 'Y is compact in X from inside'). Is then Y regular?

Our solution, Example 1, is also typical. We start with a Hausdorff non-regular space, and call it Y. Then for every closed subset, C say, of Y which is not compact, we add a

* Corresponding author.
E-mail addresses: gartside@math.pitt.edu (P.M. Gartside), glyn@maths.ox.ac.uk (A. Glyn).

PII: S0166-8641(01)00194-8

point to a superspace X which is a limit point (in X) of C, thus ensuring compactness of Y in X from inside. To do this we create a ψ-like space (in other words, related to Mrowka's construction [7] of a space from a maximal almost disjoint family of subsets of ω) 'over' Y.

The next section gives some useful lemmas. The final section contains the main results. In the sequel Y is always a subspace of the topological space X.

Useful lemmas and definitions

Let $\mathcal{A}$ be a maximal almost disjoint family of subsets of ω. Denote by $\psi(\mathcal{A})$, Mrowka's ψ-space on $\mathcal{A}$. So $\psi(\mathcal{A}) = \omega \cup \mathcal{A}$, topologised so that ω is an open discrete subspace, and a basic neighbourhood of A in $\mathcal{A}$, has the form: $\{A\} \cup (A \setminus n)$ for some $n \in \omega$. Recall that $\psi(\mathcal{A})$ is a locally compact, pseudocompact Moore space, but is not countably compact. From our perspective, a key property of $\psi(\mathcal{A})$ is that ω is dense in $\psi(\mathcal{A})$, and if S is an infinite subset of ω, then, by maximality of $\mathcal{A}$, the closure of S meets $\mathcal{A}$.

First a construction of a 'compatible' ultra-filter and maximal almost disjoint family on ω.

Lemma 1. *There is a free ultrafilter p and a maximal almost disjoint family $\mathcal{A}$, both on ω, such that given $A \in \mathcal{A}$, there is a $P \in p$ such that $P \cap A = \emptyset$.*

Proof. Let $\mathcal{A}$ be an uncountable mad family on ω.

Note that given any $A_1, \ldots, A_n \in \mathcal{A}$, $\omega \setminus (A_1 \cup \cdots \cup A_n)$ is infinite. (Otherwise, if $A \in \mathcal{A} \setminus (A_1, \ldots, A_n)$, then A must intersect some A_j on an infinite set by the pigeonhole principle; a contradiction.) Thus $\{\omega \setminus A\colon\ A \in \mathcal{A}\}$ has the finite intersection property.

Moreover $\{\omega \setminus A\colon\ A \in \mathcal{A}\} \cup \{\{n \in \omega\colon\ n \geqslant m\}\colon\ m \in \omega\}$ will also have the finite intersection property, and so forms a basis for a filter $\mathcal{F}$. $\mathcal{F}$ is contained in an ultrafilter p, which by construction must be free.

$\mathcal{A}$ and p satisfy the conditions required. □

The (well-known) machine for constructing a regular non-Tychonoff space from a Tychonoff non-normal space is used in the sequel, and is outlined below.

For further details of the following lemma see [6].

Lemma A. *Given any Tychonoff non-normal space, Z say, then there is a 'canonical' regular non-Tychonoff space, denoted X_Z say, associated with Z.*

Proof (*Sketch*). Let Z be Tychonoff with two disjoint closed subsets A and B which cannot be separated by disjoint open sets. Let $X' = (Z \times \omega) \oplus \{*\}$, where $Z \times \omega$ has the product topology, and a basic neighbourhood of $*$ is given by $\{*\} \oplus \bigcup_{n \geqslant N}(Z \times \{n\})$, for $N \in \omega$.

Identify the following points, for all $k \in \omega$:

$$\begin{array}{lll} (a, 2k) & \text{and} & (a, 2k+1) \quad (a \in A), \\ (b, 2k+1) & \text{and} & (b, 2k+2) \quad (b \in B). \end{array}$$

Call the resulting space X_Z. Since Z is regular, so is $X_Z \setminus \{*\}$. If C is closed in X_Z and does not contain $*$, then $X_Z \setminus C$ is open and must contain all the points of X_Z above a certain level n. Then the sets $U = \bigcup\{Z \times \{m\}: m < n+1\}$ and $V = \{*\} \cup \bigcup\{Z \times \{m\}: m > n+1\}$ are disjoint open sets containing C and $*$ respectively. Thus X_Z is regular.

It can be shown that there does not exist a continuous $f : X_Z \to [0, 1]$ such that $f(*) = 0$ and $f(Z \times \{0\}) = 1$. Hence X_Z is not Tychonoff. $\square$

One final well-known but useful lemma:

Lemma B. *Let X be a regular space, and let A and B be disjoint closed subsets of X, which are Lindelöf. Then A and B can be separated by disjoint open sets.*

All other undefined terms, and basic results, can be found in [5].

The results

A subspace Y is said to be *compact in X* if for every open covering of X there is a finite subfamily γ such that $Y \subseteq \bigcup \gamma$. If Y is compact in a Hausdorff space X, then Y is a regular space. Problem 1 asked if a similar result holds if Y is compact in X from inside, where Y *compact in X from inside* if every closed in X subset of Y is compact (in itself).

Example 1. There is a Hausdorff space X, with subspace Y which is compact in X from inside, such that Y is not regular.

This provides a counter-example for Problem 1.

Proof. Fix free ultrafilter p and mad family $\mathcal{A}$ as in Lemma 1. Let $Y = (\omega + 1 \times \omega) \cup \{*\}$ and $X = Y \cup (\mathcal{A} \times \omega) \cup (\mathcal{A} \times \{*\})$.

Topologise X as follows. All points of $\omega \times \omega$ are isolated. For each $n \in \omega$, (ω, n) has basic neighbourhoods of the form $\{(\omega, n)\} \cup P \times \{n\}$ for $P \in p$; and (A, n) has basic neighbourhoods $\{(A, n)\} \cup (A \setminus m) \times \{n\}$ for $m \in \omega$. (Observe that $(\omega \cup \mathcal{A}) \times \{n\}$ is $\psi(\mathcal{A}) \times \{n\}$.)

For $A \in \mathcal{A}$, basic neighbourhoods of $(A, *)$ are $\{(A, *)\} \cup \bigcup_{k \in A \setminus m} (\omega \cup \{\omega\}) \times \{k\}$. (So $\{\omega\} \times \omega \cup \mathcal{A} \times \{*\}$ is homeomorphic to $\psi(\mathcal{A})$.) And basic neighbourhoods of $*$ have the form $\{*\} \cup (\omega \times P)$ for $P \in p$.

Y is not regular because if $U = \{*\} \cup \omega \times \omega$ (an open set), then any basic neighbourhood of $*$ has closure meeting $\{\omega\} \times \omega$.

Checking that X is Hausdorff requires checking various cases, which are either straightforward or rely on the properties of $\mathcal{A}$ and p given by Lemma 1. For example, one case is that of separating (ω, n) from some (A, n). Then there is a $P \in p$ such that $P \cap A = \emptyset$, and the obvious basic neighbourhoods work.

It remains to show if S is an infinite subset of Y then $\overline{S} \cap (X \setminus Y) \neq \emptyset$. There are three cases depending on where S lies.

Case (i). There is an n such that $S \cap (\omega \times \{n\})$ is infinite.

Then there is an $A \in \mathcal{A}$ such that every neighbourhood of (A, n) meets S. (Let $S' = \{m \in \omega\colon (m, n) \in S\}$. If $S' \cap A$ were finite for all $A \in \mathcal{A}$ then $\mathcal{A}$ would not be maximal.)

Case (ii). $S \cap (\{\omega\} \times \omega)$ infinite.

Then there is an A in $\mathcal{A}$ such that every neighbourhood of $(A, *)$ meets $S \cap (\{\omega\} \times \omega)$.

Case (iii). $\{m\colon S \cap (\omega \times \{m\}) \neq \emptyset\}$ is infinite.

Then there is an A in $\mathcal{A}$ such that every neighbourhood of $(A, *)$ meets $S \cap (\omega \times \omega)$. □

Lemma 2. *The example given above is also Urysohn* (*that is, for each $x, y \in X$, $x \neq y$, there are open neighbourhoods U, V containing x, y, respectively, such that $\overline{U}^X \cap \overline{V}^X = \emptyset$*).

Thus Example 1 also provides a counter-example for Problem 5. In contrast, it is known (see [3]) that if Y is compact in a Urysohn space X, then Y is a Tychonoff space.

Proof. Note that basic open neighbourhoods about points of the form (m, n) (where $m, n \in \omega$), (A, n) (where $A \in \mathcal{A}, n \in \omega$), and $(A, *)$ (for $A \in \mathcal{A}$) are in fact clopen. For $n \in \omega$, (ω, n) has basic open neighbourhood $\{\omega, n\} \cup (P \times \{n\})$ for some $P \in p$, and $\overline{\{\omega, n\} \cup (P \times \{n\})}^X = \{\omega, n\} \cup (P \times \{n\}) \cup \{(A, n)\colon A \in \mathcal{A}, |P \cap A| = \aleph_0\}$. Also, $\{*\}$ has basic open neighbourhood $\{*\} \cup (\omega \times P)$ for some $P \in p$, and $\overline{\{*\} \cup (\omega \times P)}^X = \{*\} \cup (\omega \times P) \cup (\{\omega\} \times P) \cup (\mathcal{A} \times P)$.

Checking that X is Urysohn again requires checking various cases, using repeatedly the properties of $\mathcal{A}$ and p given by Lemma 1. □

A subspace Y is *regular in X* if for each $y \in Y$ and each closed in X subset C of X such that $y \notin C$, there are open in X disjoint subsets U, V of X such that $y \in U$ and $C \cap Y \subseteq V$. It was shown in [4] that a Hausdorff space Y is regular in every larger Hausdorff space if and only if Y is compact. Problem 2 asked whether a similar result holds if the condition of Y regular in X is replaced by Y internally regular in X, where Y is *internally regular in X* if for each $x \in X$ and every subset C of Y closed in X, where $x \notin C$, there are open in X disjoint subsets U, V of X such that $x \in U$ and $C \subseteq V$.

Theorem 1. *Let Y be a Hausdorff space internally regular in every larger Hausdorff space. Then Y is compact.*

This gives a positive answer to Problem 2. Arhangel'skii has informed the authors (private communication) that Jack Porter has, independently, solved Problem 2.

Proof. We prove the contrapositive:

Given a Hausdorff non-compact space Y, there is a Hausdorff space X, containing Y, in which Y is not internally regular.

Fix a proper closed subset C_0 of Y, and family, $\mathcal{C}$ say, of closed non-empty subsets of Y (including C_0), all contained in C_0, which is closed under finite intersections, but has empty intersection (so $\mathcal{C}$ 'witnesses' non-compactness of Hausdorff Y).

Choose $y_0 \in Y \setminus C_0$. We aim to extend Y to a Hausdorff space X, so as to make C_0 closed in X, but y_0 and C_0 cannot be separated by disjoint open subsets of X.

Let $X = Y \oplus \bigoplus_{C \in \mathcal{C}} (C \times \omega)$, and topologise X as follows. $Y \setminus (C_0 \cup \{y_0\})$ is an open subspace. All points of $(C_0 \times \omega)$ are isolated. A basic open neighbourhood of y_0 has the form $U \oplus \bigoplus\{C' \times \omega\colon\ C' \in \mathcal{C},\ C' \subseteq C\}$ for U an open neighbourhood in $Y \setminus C_0$ of y_0 and some C in $\mathcal{C}$. And a basic open neighbourhood of x in C_0 has the form $V \oplus \bigoplus\{(V \cap C) \times (\omega \setminus n)\colon\ V \cap C \neq \emptyset,\ C \in \mathcal{C}\}$ for V an open neighbourhood of x in Y with $y_0 \notin V$, and some $n \in \mathbb{N}$.

To show that this does define a topology on X, need to check that finite intersections give open sets of the required form. The one case which needs to be checked is as follows:

Let $W = U \oplus \bigoplus\{C' \times \omega\colon\ C' \in \mathcal{C}, C' \subseteq C\}$ be a basic open neighbourhood about y_0, and let $Z = V \oplus \bigoplus\{(V \cap D) \times (\omega \setminus n)\colon\ V \cap D \neq \emptyset,\ D \in \mathcal{C}\}$ be a basic open neighbourhood about $x \in C_0$.

Then $W \cap Z = (U \cap V) \oplus \bigoplus\{(V \cap (C' \cap D)) \times (\omega \setminus n)\colon\ V \cap (C' \cap D) \neq \emptyset,\ C', D \in \mathcal{C},\ C' \subseteq C\} = (U \cap V) \oplus \bigoplus\{((V \cap C) \cap D) \times (\omega \setminus n)\colon\ D \in \mathcal{C}\}$, which is open since $U \cap V \subseteq Y \setminus (C_0 \cup \{y_0\})$.

Moreover, Y has the subspace topology in X, and C_0 is closed in X.

Claim 1. *Every basic open neighbourhood of y_0 has closure (in X) meeting the set C_0. Thus Y is not internally regular in X.*

Let $W = U \oplus \bigoplus\{C' \times \omega\colon\ C' \in \mathcal{C},\ C' \subseteq C\}$ be a basic open neighbourhood of y_0. Take $x \in C \subseteq C_0$. Let $Z = V \oplus \bigoplus\{(V \cap D) \times (\omega \setminus n)\colon\ V \cap D \neq \emptyset,\ D \in \mathcal{C}\}$ be a basic open neighbourhood of x. Now $V \cap C \neq \emptyset$ since $x \in C$. Choose $m \in \mathbb{N}$ such that $m > n$. Then $(x, m)_C \in ((V \cap C) \times (\omega \setminus n))_C \cap (C \times \omega)_C$, and so $W \cap Z \neq \emptyset$. Hence $x \in \overline{W}$, and so $C_0 \cap \overline{W} \neq \emptyset$.

Claim 2. *X is Hausdorff.*

Take y_0 and $x \in C_0$. Y is Hausdorff, so there are open in Y disjoint subsets U, V such that $y_0 \in U$ and $x \in V$. Since $\bigcap_{C \in \mathcal{C}} C = \emptyset$, there is a $C \in \mathcal{C}$ such that $x \notin C$. Let $W = Y \setminus C$, open in Y with $x \in W$. Set $Z = V \cap W$ (note $Z \cap C = \emptyset$). Then $x \in A = Z \oplus \bigoplus\{(Z \cap D) \times (\omega \setminus n)\colon\ Z \cap D \neq \emptyset,\ D \in \mathcal{C}\}$, for some $n \in \mathbb{N}$, and $y_0 \in B = U \oplus \bigoplus\{C' \times \omega\colon\ C' \in \mathcal{C},\ C' \subseteq C\}$, where by construction A, B are disjoint open sets in X, as required.

The remaining cases follow from the topology defined on X, remembering that Y (as a subspace) is Hausdorff. □

Example 2. There is a T_3 space X, with dense subspace Y, where Y is compact in X from inside, but Y is not Tychonoff.

This provides a counter-example to Problem 3.

Proof. Let $L' = I^2$ with the Tangent Disc topology, so that points in $I \times (0,1]$ have their usual (Euclidean) neigbourhoods and basic open neighbourhoods of an $(x,0)$ have the form $U(x,n)$, where $U(x,n)$ is $\{(x,0)\}$ along with an open disc centered at $(x,1/n)$ and radius $1/n$. Let $Q' = (\mathbb{Q} \cap I) \times \{0\}$, and $P' = (I \setminus \mathbb{Q}) \times \{0\}$. Note that P', Q' are disjoint closed subsets of L' which cannot be separated by disjoint open sets. Let $\mathcal{A}$ be a mad family on Q', and $\mathcal{B}$ be a maximal almost disjoint family of countably infinite subsets of P'. Let $L = L' \oplus \mathcal{A} \oplus \mathcal{B}$, topologised so that L' is an open subspace and basic open neighbourhoods of an A in $\mathcal{A}$ and a B in $\mathcal{B}$ are (respectively) of the form:

$$\{A\} \oplus \bigcup_{x \in A \setminus F} U(x, n_x) \quad \text{for finite } F \subseteq A \text{ and } n_x \in \mathbb{N},$$

$$\{B\} \oplus \bigcup_{x \in B \setminus G} U(x, m_x) \quad \text{for finite } G \subseteq B \text{ and } m_x \in \mathbb{N}.$$

Note that $Q = Q' \cup \mathcal{A}$ and $P = P' \cup \mathcal{B}$ are disjoint closed sets in L, which cannot be separated by disjoint open sets.

Since L' is certainly T_3, we can separate in L' any pair of *countable* subsets of $I \times \{0\}$ (using Lemma B), and we can deduce that L is T_3. Further L has the following key property $(*)$: if closed $C \subseteq L$, then either $C \cap (I \times \{0\})$ is finite—and C is compact, or $C \cap (I \times \{0\})$ is infinite—and (depending on whether $C \cap Q'$ or $C \cap P'$ is infinite) C meets $\mathcal{A}$ or $\mathcal{B}$ (or both).

Let $X = X_L$, constructed using L, P, and Q as in Lemma A, and let $Y = X_{L'}$, constructed using L', P', and Q'. Note that Y is a dense subspace of X.

Then X is T_3, but Y is not Tychonoff as the closed set $P' \times \{0\}$ and the point $\{*\}$ cannot be separated by a continuous real valued function. ("$Y = X_{L'}$", using the notation of Lemma A.)

It remains to show Y is compact in X from inside. But if C is a closed subset of X contained in Y, then by the key property $(*)$ C is compact in each 'level' of X, so either is a finite union of compacta (and hence compact) or a sequence of compacta converging to the single point $*$, and thus, again, compact. □

In the above example, X is Hausdorff, and so 'Y compact in X from inside' implies 'Y is internally normal in X' (see definition below). Thus the above example also provides a counter-example to Problem 7.

A subspace Y is *normal in X from inside* if every closed in X subspace of Y is normal (in itself). While Y is *internally normal in X* if for each pair A, B of closed in X disjoint subsets of Y, there are disjoint subsets U, V of X such that $A \subseteq U$ and $B \subseteq V$.

Example 3. There is a space X and a subspace Y of X such that Y is normal in X from inside, but Y is not internally normal in X.

Proof. Let X be the tangent disc space, and Y the real axis. □

Example 3 answers Problem 4 as originally set, but Arhangel'skii (private communication) has confirmed that he meant for Y to be dense in X. Example 4 answers this variant of Problem 4.

Example 4. There is a Tychonoff space X and a dense subspace Y of X such that Y is normal in X from inside, but Y is not internally normal in X.

Proof. Let $\mathcal{A}$ be a mad family and p a free ultrafilter on ω, as in Lemma 1.

Denote by $\psi(\mathcal{A}, p)$ the space with underlying set $\omega \cup \{p\} \cup \mathcal{A}$, topologised as follows: All points of ω are isolated. A basic neighbourhood of $A \in \mathcal{A}$ is given by $\{A\} \cup (A \setminus n)$ for some $n \in \mathbb{N}$, and an open neighbourhood of p is given by $\{p\} \cup P$ for some $P \in p$. Let L be the subspace of $\omega_1 + 1$ consisting of the isolated points of ω_1 and the end point ω_1. Let $R = \omega \cup \{p\}$ (with the usual topology, inherited from $\beta\omega$).

Let $Y = L \times R \setminus \{(\omega_1, p)\}$ and $X = (L \setminus \{\omega_1\}) \times \psi(\mathcal{A}, p) \cup \{\omega_1\} \times \omega$; where X is topologised as a subspace of $L \times \psi(\mathcal{A}, p)$.

X Tychonoff: L and $\psi(\mathcal{A}, p)$ are both zero-dimensional and Hausdorff ($\psi(\mathcal{A}, p)$ is Hausdorff using Lemma 1), and so $L \times \psi(\mathcal{A}, p)$ is zero-dimensional Hausdorff. Thus X, as a subspace of a Tychonoff space, is itself Tychonoff.

Y is not internally normal in X: Clearly Y is dense in X; and $(L \setminus \{\omega_1\}) \times \{p\}$ and $\{\omega_1\} \times \omega$ are disjoint, closed in X, contained in Y, and cannot be separated in Y (and, *a fortiori* not in X). Hence Y is not internally normal in X.

Y is normal in X from inside: Take $C \subseteq Y$, C closed in X. Since the only non-isolated (in Y) points of Y are on the 'top edge' (α, p) (for $\alpha \in L \setminus \{\omega_1\}$) and the 'right edge' (ω_1, n) (for $n \in \omega$), we only need worry about separating $C \cap (L \setminus \{\omega_1\} \times \{p\})$ and $C \cap (\{\omega_1\} \times \omega)$ in C.

For each $\alpha \in (L \setminus \{\omega_1\})$, $C \cap (\{\alpha\} \times (\omega \cup \{p\}))$ is finite (otherwise C would have a limit point in $\{\alpha\} \times \mathcal{A}$, since $\mathcal{A}$ a mad family—contradicting C closed in X and contained in Y). Hence $C \cap ((L \setminus \{\omega_1\}) \times \{p\})$ is open (as well as closed) in C. So we can certainly separate it from $C \cap (\{\omega_1\} \times \omega)$. $\square$

A subspace Y is *normal in* (respectively *weakly normal in*) X if whenever C and D are disjoint closed subsets of X, then there are disjoint U and V separating $C \cap Y$ and $D \cap Y$ where U and V are open in X (respectively open in Y). Clearly 'normal in' implies 'weakly normal in', and if Y is dense in X, then it is easy to see that the converse holds as well. In the example below we make Y dense in X, and ensure Y is normal in X by forcing any pair of disjoint closed subsets of Y which cannot be separated in Y to have point in common in their closure in X.

Example 5. ($\mathrm{MA} + \aleph_2 < 2^{\aleph_0}$) There is a T_3 space X, with a dense subspace Y, so that Y is normal in X, but Y is not Tychonoff.

Hence there is a consistent counter-example to Problem 6 (which was also stated in [1] as Problem 33).

Proof. Let $R = \omega_1 + 1$, with order topology refined so that all $\alpha < \omega_1$ are isolated, and $S = \omega_2 + 1$, with order topology refined so that all $\alpha < \omega_2$ are isolated. Let $L = R \times S \setminus \{(\omega_1, \omega_2)\}$. Then $A = (\omega_1 + 1) \times \{\omega_2\} \setminus \{(\omega_1, \omega_2)\}$ and $B = \{\omega_1\} \times (\omega_2 + 1) \setminus \{(\omega_1, \omega_2)\}$ are pairwise disjoint closed subsets of L which cannot be separated by disjoint open sets. Let $Y = X_L$—the T_3 non-Tychonoff space built from L, A and B, as given by Lemma A.

We now construct a T_3 space X in which Y is a dense (weakly) normal subspace. To do so requires some combinatorics. Since we are assuming Martin's Axiom and that $\aleph_2 < 2^{\aleph_0}$, we have that $\aleph_1^{\aleph_1}.\aleph_2^{\aleph_1} = 2^{\aleph_0}$. Hence we can list $[\omega_1]^{\aleph_1} \times [\omega_2]^{\aleph_1} = \{(P_\alpha, Q_\alpha)\}_{\alpha \in 2^{\aleph_0}}$. Again by our assumptions we can list a maximal almost disjoint family of countably infinite subsets of ω_1 as $\mathcal{A} = \{A'_\alpha\}_{\alpha \in 2^{\aleph_0}}$, and a maximal almost disjoint family of countably infinite subsets of ω_2 as $\mathcal{B} = \{B'_\alpha\}_{\alpha \in 2^{\aleph_0}}$. (In each case the listing is without repetitions.)

Recourse through the (P_α, Q_α)s picking the first $A^0_\alpha = A'_{\alpha'}$ and second $A^1_\alpha = A'_{\alpha''}$ in $\mathcal{A} \setminus \{A^i_\beta\colon\ \beta < \alpha,\ i = 0, 1\}$ and the first $B^0_\alpha = B'_{\alpha'}$ and second $B^1_\alpha = B'_{\alpha''}$ in $\mathcal{B} \setminus \{B^i_\beta\colon\ \beta < \alpha,\ i = 0, 1\}$ such that, for $i = 0, 1$, $|A^i_\alpha \cap P_\alpha| = \aleph_0$ and $|B^i_\alpha \cap Q_\alpha| = \aleph_0$. To see this is possible, pick countably infinite $G \subseteq P_\alpha$, observe (by maximality of $\mathcal{A}$) $\{G \cap A'_\beta\colon\ |G \cap A'_\beta| = \aleph_0,\ \beta \in 2^{\aleph_0}\}$ is a maximal almost disjoint family on G, and, since MA holds, this family must have size $2^{\aleph_0}$; as $\alpha < 2^{\aleph_0}$ we have many choices for A^0_α and A^1_α (and similarly for $Q_\alpha, \mathcal{B}, B^0_\alpha, B^1_\alpha$).

Let $M^i = \{(A^i_\alpha, B^i_\alpha)\colon\ \alpha \in 2^{\aleph_0}\}$ for $i = 0, 1$. Fix n in ω, and let $i = n \bmod 2$. Define $L_n = (R \times S) \setminus \{(\omega_1, \omega_2)\} \cup M^i$, where $(R \times S) \setminus \{(\omega_1, \omega_2)\}$ is an open subspace, and a basic open neighbourhood of $(A^i_\alpha, B^i_\alpha) \in M^i$ is given by:

$$\big\{\big(A^i_\alpha, B^i_\alpha\big)\big\} \cup \bigcup_{x \in A^i_\alpha \setminus F} \big(\{x\} \times (z_x, \omega_2]\big) \cup \bigcup_{y \in B^i_\alpha \setminus G} \big((z_y, \omega_1] \times \{y\}\big)$$

for finite $F \subseteq A^i_\alpha$, finite $G \subseteq B^i_\alpha$, some $z_x \in \omega_2$, and some $z_y \in \omega_1$.

Let $X' = \{*\} \oplus \{\bigcup_{n \in \mathbb{N}} (L_n \times \{n\})\}$, where each $L_n \times \{n\}$ is an open subspace, and a basic open neighbourhood of $\{*\}$ is given by $\{*\} \oplus \bigcup_{n \geqslant N} (L_n \times \{n\})$, for some $N \in \omega$.

Identify the following points, for all $k \in \omega$:

$$\begin{array}{lll} (a, 2k) & \text{and} \quad (a, 2k+1) & (a \in A), \\ (b, 2k+1) & \text{and} \quad (b, 2k+2) & (b \in B). \end{array}$$

Call the resulting space X, and note that Y is a subspace of X.

That X is T_3 is ensured by the fact that countable subsets of R or S are closed, and the fact that if $\alpha \neq \alpha'$ or $i \neq j$, then $A^i_\alpha \cap A^j_{\alpha'}$ and $B^i_\alpha \cap B^j_{\alpha'}$ are finite. Clearly Y is dense in X. So it remains to show Y is (weakly) normal in X.

Let C and D be disjoint closed subsets of X. If $*$ is in C then D must meet only a finite number of levels of Y (and symmetrically). So it suffices to show C and D can be separated at each level individually. But for any particular level, say the nth, separating C and D restricted to that level is only a problem if $C_n = C \cap (A \times \{n\})$ and $D_n = D \cap (B \times \{n\})$ are uncountable (or symmetrically). However (taking a subset of size $\aleph_1$ if necessary) $(C_n, D_n) = (P_\alpha, Q_\alpha)$ for some α, and then $(A^i_\alpha, B^i_\alpha, n) \in M^i \times \{n\} \subseteq X \setminus Y$ (where $i = n \bmod 2$) is a common limit point of C_n and D_n—and this contradicts disjointness of 'closed and disjoint' C and D in X. □

A real valued function f on a space X is said to *Y-continuous*, on a subspace Y, if it is continuous at each point of Y. A subspace Y is *realnormal in X* if for all closed disjoint A and B in X, there is a Y-continuous $f : X \to \mathbb{R}$ such that $f(A \cap Y) \subseteq \{0\}$ and $f(B \cap Y) \subseteq \{1\}$. In his survey article [1] Arhangel'skii asked the following question.

Problem 30. Let X be regular, Y normal in X and dense in X. Is then Y realnormal in X? What if we drop 'dense'? What if we assume X to be Tychonoff?

Since if Y is realnormal in T_1 X then Y is clearly Tychonoff, our example above answers, at least consistently, the first part of Arhangel'skii's question in the negative (and makes the second part redundant). The following example is a consistent counter-example to the last part of Problem 30.

Example 6. ($\mathrm{MA} + \aleph_2 < 2^{\aleph_0}$) There is a Tychonoff space X^* with dense subspace Y^*, normal in X^* but not realnormal in X^*.

Proof. Our space is a subspace of the previous example and we use the notation established there. Write Y^* for levels $0, 1$ and 2 of Y and let $X^* = Y^* \cup (\bigcup_{n=0,1,2} M^{(n \bmod 2)} \times \{n\})$. Note that X^* is Tychonoff, Y^* is an open dense subspace, and (as in the previous example) Y^* is normal in X^*.

Write A_2 for the copy of A in level 2, and let A^* be the union of A_2 along with $M^2 \times \{0\}$. Write B_0 for the copy of B in level 0, and let B^* be B_0 union $M^0 \times \{0\}$. (Of course, $A^* \cap Y^* = A_2$ and $B^* \cap Y^* = B_0$.) Then A^* and B^* are disjoint closed subsets of X^*. Suppose, for a contradiction, that Y^* were realnormal in X^*. Then there would be a continuous $f : Y^* \to \mathbb{R}$ such that $f(A_2) = \{0\}$ and $f(B_0) = \{1\}$. And hence $U = f^{-1}(-\infty, 1/3)$ and $V = f^{-1}(2/3, \infty)$ would be disjoint, open in Y^*, $A_2 \subseteq U$, $B_0 \subseteq V$, $\overline{U} \cap \overline{V} = \emptyset$ and $\overline{U}$, $\overline{V}$ would be separated by disjoint open (in Y^*) $f^{-1}(-\infty, 1/2)$ and $f^{-1}(1/2, \infty)$.

But any open (in Y^*) $U' \supseteq A_2$ must have its Y^*-closure meeting the copy of B in level 1 in a set of cardinality $\aleph_2$. And any open (in Y^*) $V' \supseteq B_0$ must have its Y^*-closure meeting the copy of A in level 1 in a set of cardinality $\aleph_1$. However any subset of A of cardinality $\aleph_1$ and any subset of B of cardinality $\aleph_2$ (in level 1) cannot be separated by disjoint open sets (in Y^*)—contradiction. $\square$

A space X is *normal on Y* if for every pair S, T of subsets of Y with disjoint closures in X there are disjoint open sets in X separating those closures. Further, X is *collectionwise normal on Y* if for every family $\{S_\alpha\}_{\alpha \in A}$ of subsets of Y whose closures (in X) form a discrete family of closed sets in X, there is a discrete, in X, family of open sets separating those closures. X is *densely* (*respectively collectionwise*) *normal* if there is a dense subset of X on which X is (respectively collectionwise) normal. For any cardinal κ, X is *densely κ-collectionwise normal* if it has a dense subspace on which collectionwise normality occurs for collections of size no more than κ. By way of comparison, let us observe that every normal space is $\aleph_0$-collectionwise normal.

Example 7. There is a first countable, locally compact Tychonoff space which is densely normal, but not densely $\aleph_0$-collectionwise normal.

This provides a counter-example for Problems 9 and 10.

Proof. Let $\mathbb{E}$ denote the even integers, and $\mathbb{O}$ the odd integers. Let $\mathcal{A}_0 = \{\mathbb{E} \times \{n\}\colon n \in \omega\}$. Let $\mathcal{A}_1$ be a maximal almost disjoint family of subsets of $\mathbb{E} \times \omega$ extending $\mathcal{A}_0$. Let $\mathcal{A} = \mathcal{A}_1 \setminus \mathcal{A}_0$. The underlying set of the space X is $\{(\omega + 1) \times \omega\} \oplus \mathcal{A}$; topologised so that $(\omega + 1) \times \omega$ is an open subspace and has the product topology, while $\{\mathbb{E} \times \omega\} \oplus \mathcal{A}$ is an open subspace with the standard ψ-space topology. Let $Y = \omega \times \omega$.

Then X is first countable, locally compact, and Hausdorff, and hence also Tychonoff.

We first show X is not densely $\aleph_0$-collectionwise normal. So let D be any dense subspace of X. Note that $Y \subseteq D$. For $n \in \omega$, define $C_n = (\mathbb{O} \cup \{\omega\}) \times \{n\} = \overline{\mathbb{O} \times \{n\}}^X$. Then each C_n is closed. and discrete in X. For each A in $\mathcal{A}$, $A \cap (\omega \times \{n\}) \subseteq \mathbb{E} \times \{n\}$, and so misses $\mathbb{O} \times \{n\}$. Thus the family $\{C_n\colon n \in \mathbb{N}\}$ is also discrete.

However the family $\{C_n\colon n \in \mathbb{N}\}$ cannot be separated in X by a discrete family of open sets. Suppose they could be, and $C_n \subseteq U_n$ where the $\{U_n\colon n \in \mathbb{N}\}$ form a discrete collection of open sets. Then for each n in ω, pick even N_n such that $(N_n, n) \in U_n$. Let $S = \{(N_n, n)\colon n \in \omega\}$. This is an infinite subset of $\mathbb{E} \times \omega$. Hence by maximality of $\mathcal{A}_1$, there is an A in $\mathcal{A}$ such that $A \cap S$ is infinite. In which case every open neighbourhood of A meets U_n for infinitely many n, contradicting discreteness.

The above proves that X is indeed not densely $\aleph_0$-collectionwise normal.

It remains to show that X is normal on Y, in other words to prove that if S and T are disjoint infinite subsets of Y then either we can separate $\overline{S}^X$ and $\overline{T}^X$, or $\overline{S}^X \cap \overline{T}^X \neq \emptyset$.

Note that as points of Y are isolated it is sufficient to consider only what happens for points in $X \setminus Y$. There are various cases to consider. If, for some n, the set $S \cap (\omega \times \{n\})$ is infinite, then either $T \cap (\omega \times \{n\})$ is infinite, in which case the closures of S and T meet, or $T \cap (\omega \times \{n\})$ is finite, in which case we can separate the two closures restricted to $(\omega + 1) \times \{n\}$ without difficulty.

For the remainder of the argument we assume the second option above holds for all n. Now other possible limit points of S and T, are respectively in the closures in X of, $S_e = S \cap (\mathbb{E} \times \omega)$ and $T_e = T \cap (\mathbb{E} \times \omega)$.

Two cases arise. If for all $A \in \mathcal{A}$ such that $A \cap S_e$ is infinite, $T_e \cap A$ is finite and for all $A \in \mathcal{A}$ such that $A \cap T_e$ is infinite, $S_e \cap A$ is finite, then we can separate the relevant parts of the closures of S and T with:

$$S_e \cup \bigcup \bigl\{(A \setminus T_e) \cup \{A\}\colon A \in \mathcal{A},\ |A \cap S_e| = \aleph_0\bigr\} \quad \text{and}$$
$$T_e \cup \bigcup \bigl\{(A \setminus S_e) \cup \{A\}\colon A \in \mathcal{A},\ |A \cap T_e| = \aleph_0\bigr\}.$$

If the above case fails to hold, then there must be an A in $\mathcal{A}$ such that both $A \cap S_e$ and $A \cap T_e$ are infinite. And this means A is in the closure of both S_e and T_e—hence we have that the closures of S and T are not disjoint. $\square$

A subspace Y is *bounded in* X if every continuous real-valued function on X is bounded on Y. And Y is *countably compact in* X if for each countable open covering of X, there is a finite subfamily covering Y.

Example 8. There is a Tychonoff space X, with a dense subspace Y, such that Y is internally normal in X and bounded in X, but not countably compact in X.

This gives a counter-example to Problem 17.

Proof. Let $\mathcal{A} = \{A_\alpha\}_{\alpha\in\kappa}$ be a mad family on ω. Let $X = \psi(\mathcal{A})$ (ψ-space from $\mathcal{A}$). Then X is Tychonoff and pseudocompact. Hence all subspaces of X are bounded in X. Let $Y = \omega \cup \{A_n\}_{n\in\omega}$, which is dense in X.

Let A, B be closed in X disjoint subsets of Y. Since A, B are both Lindelöf (they are both countable), and X is T_3, by Lemma B they can be separated by disjoint open sets in X. Thus Y is internally normal in X.

It remains to show Y is not countably compact in X. Let

$$\mathcal{U} = \bigl\{\{A_\alpha\}_{\alpha\geqslant\omega} \cup \omega\bigr\} \cup \bigl\{\{A_n\} \cup \omega\bigr\}_{n\in\omega}.$$

Then $\mathcal{U}$ is a countable open cover of X. But no finite subcollection covers all of $\{A_n\}_{n\in\omega}$ (each member of the cover meets the latter set in either zero or one point(s)). $\square$

The following remarkable example is due to Shakmatov [8].

Example. There is a dense pseudocompact subspace, Y say, of $I^{\mathfrak{c}}$ such that every countable subspace is closed and discrete in Y and any two countable subsets have disjoint closures in $I^{\mathfrak{c}}$.

Example 9. There is a Tychonoff space X with dense pseudocompact subspace Y, such that Y is normal in X from inside, but Y is not countably compact in X.

This gives a counter-example to Problem 19.

Proof. Let Y be Shakmatov's space. Write Y as the disjoint union of A and B where A is some countably infinite subset of Y.

Let C be any countably infinite subset of B. Then $\overline{C}^{I^{\mathfrak{c}}}$ is not contained in Y, for otherwise $\overline{C}^{I^{\mathfrak{c}}}$ would be an infinite discrete compact subset of Y ($I^{\mathfrak{c}}$ compact). So for each such C, choose $x_C \in \overline{C}^{I^{\mathfrak{c}}} \setminus Y$.

Let X be Y along with all these points. Then Y is pseudocompact, and dense in Tychonoff X. By the properties of Y, none of the x_C are in the closure of A in $I^{\mathfrak{c}}$, and so A is a countably infinite subset of Y which is closed and discrete in X. Hence Y is not countably compact in X.

To show that Y is normal in X from inside, it suffices (since countable T_3 spaces are normal) to show that every uncountable subset of Y has closure meeting $X \setminus Y$. But if S is such an uncountable subset, then it contains some countably infinite subset, C say, of B. And x_C was specifically chosen to be a point of X in the closure of C, outside Y. $\square$

References

[1] A.V. Arhangel'skii, Relative topological properties and relative topological spaces, Topology Appl. 70 (1996) 87–99.
[2] A.V. Arhangel'skii, Some questions and references on relative topological properties, Part 1, Topology Atlas, June 15 (2000), http://at.yorku.ca/i/a/a/i/04.htm.
[3] A.V. Arhangel'skii, I.V. Yaschenko, Relatively compact spaces and separation properties, Comm. Math. Univ. Carolin. 37 (1996) 343–348.
[4] A.V. Arhangel'skii, J. Tartir, A characterization of compactness by a relative separation property, Questions Answers Gen. Topology 14 (1996) 49–52.
[5] R. Engelking, General Topology, Heldermann, Berlin, 1989.
[6] F.B. Jones, Hereditarily separable, non-completely separable spaces, in: Topology Conference (Virginia Polytechnic Inst. and State Univ., 1973), Lecture Notes in Mathematics, Springer, Berlin, 1974, pp. 149–152.
[7] S. Mrowka, On completely regular spaces, Fund. Math. 41 (1954) 105–106.
[8] D.B. Shakmatov, A pseudocompact Tychonoff space all countable subsets of which are closed and C^*-embedded, Topology Appl. 22 (1986) 139–144.